Lecture 1

Let $\Omega \subseteq \mathbb{C}^n$ be a domain/region (open connected subset).

<u>Def 1</u> A function $f: \Omega \to \mathbb{C}$ is <u>holomorphic</u> if (i) f is $\mathcal{C}^1$ (cont. partial derivatives)

(ii) $\frac{\partial f}{\partial \bar{z}_j} = 0$, $j = 1, \dots, n$.

Here $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j}\right)$, and we use coordinates $z_j = x_j + i y_j$ in $\mathbb{C}^n$, $z = (z_1, \dots, z_n)$.

<u>Note</u>. This is of course consistent w/ the one variable case. Writing $f = u + iv$ and the equation $\frac{\partial f}{\partial \bar{z}} = 0$ in real and imaginary parts yields the Cauchy-Riemann equations
$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

Many properties of holomorphic functions in $\mathbb{C}^n$ parallel those in 1D. We summarize.

① <u>Cauchy's Integral formula</u>.

A polydisk of multiradius $r=(r_1,\dots,r_n)$ and center $a=(a_1,\dots,a_n)$ is a product of 1D disks $D_1,\dots,D_n\subseteq\mathbb{C}$:

$$D^n = D_1\times\dots\times D_n =$$

$$= \{z\in\mathbb{C}^n : |z_j-a_j|<r_j\}$$

The n-torus $\partial D_1\times\dots\times\partial D_n$ is called the distinguished boundary $\partial_0 D^n$.

Note the boundary of a domain is typically a $2n-1$ real dimension "set".
The distinguished boundary of D^n is an n-dimensional real mfld.

Thm 1. Let $\overline{D^n} \subseteq \Omega$ and $f: \Omega \to \mathbb{C}$ a holomorphic fcn. Then, for $z \in D^n$,

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 D^n} \frac{f(\zeta)}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)} d\zeta_1 \cdots d\zeta_n .$$

Rem. (i) The integral can be parametrized $\zeta_j = a_j + r_j e^{it_j}$ and be written

$$f(z) = \frac{1}{(2\pi i)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{f(a_1 + r_1 e^{it_1}, \dots)\, i^n r_1 \cdots r_n e^{i(t_1 + \dots + t_n)}}{\prod_{j=1}^n (r_j e^{it_j} + a_j - z_j)} dt_1 \dots dt_n$$

(ii) A consequence is that f is C^∞.

(iii) It suffices that f is continuous and holomorphic in each variable separately. It also follows that such f is holomorphic. In other words, a separately holomorphic function is actually holomorphic.